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A Complete Study on a Substantial Ecological Model

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ABSTRACT

The paper is aimed to explore a substantial ecosystem by conceptualizing the model with a Prey-Ammensal, a Predator and an Enemy to the Prey-Ammensal.Limited resources are considered for all species. Perturbation analysis is carried out for identifying the existence of the ecological model. The studies of local stability and global stability established the behavior and nature of the ecosystem. Diffusion analysis and Stochastic Analysis are employed. The Homotopy perturbation approach is used to evaluate the series solutions. Whenever possible, the appropriate graphs are illustrated.

Key words:
Local Stability, Global Stability, HPM, Gaussian White Noise, Diffusion Analysis

1. Introduction

The analysis of the diversity and distribution of various species using the same resources in the same ecosystem may also be referred as ecology. The Local stability and Global stability of various ecological models were established by K.V.L.N.Acharyulu and N.ch. PattabhiRamacharyulu in multifarious aspects [5-9]. Eminent Scholars, Mathematicians [1-4] and Lotka A.J [10] defined and executed useful principles to identify the nature of various ecological models.

1.1 Notations Adopted for the Significant Ecosystem:

In this model, $N_1(t)$ represents Prey-Ammesnal species population strength with the natural growth rate a_1.$N_2(t)$ stands for the predator's population strength striving of the Prey-Ammensal (N_1) with the natural growth rate a_2.$N_3(t)$ denotes the Population Strength of the enemy to the Prey-Ammesnal(N_1) with the natural growth rate a_3.a_{ii} refers the rate of decrease of N_i due to insufficient resources of N_i,i =1,2,3.a_{12} can be represented as the rate of decrease of the Prey-Ammesnal(N_1) due to inhibition by the predator (N_2).a_{13} refers the rate of decrease of the Ammensal (N_1) due to competitive inhibition from enemy (N_3).a_{21} stands for the rate of increase of the predator (N_2) due to its successful attacks on the Prey-Ammesnal species (N_1).The carrying capacities of N_i ,i = 1, 2, 3 are represented by K_i : a_i/a_{ii}.Theco-efficient of Prey-Ammensalism is denoted by $\alpha= a_{13}/a_{11}$.The co-efficient of Prey-Ammesnal suffering rate is represented by $P=a_{12}/a_{11}$.The co-efficient of predator consumption of the Prey-Ammesnal is labeled by $Q=a_{21}/a_{22}$

All these variables and considered parameters are assumed as non-negative.

2. Basic Equations

The equations for the significant ecosystem are constituted as

(i) The rate of growth equation for Prey-Ammensal species (N_1):

$$\frac{dN_1}{dt} = a_{11}N_1(K_1- N_1- PN_2 - \alpha N_3) \qquad (2.1)$$

(ii) The rate of growth equation for predator (N_2):

$$\frac{dN_2}{dt} = a_{22} N_2 (K_2 - N_1 + Q N_1) \qquad (2.2)$$

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(iii) The rate of growth equation for enemy (N_3):

$$\frac{dN_3}{dt} = a_{33}N_3(K_3 - N_3) \quad (2.3)$$

Here The co-existent state $E^*\left(\overline{N_1},\overline{N_2},\overline{N_3}\right)$ exists at

(viii) $\overline{N_1}=\frac{K_1-PK_2-\alpha K_3}{1+PQ}$; $\overline{N_2}=\frac{QK_1+K_2-Q\alpha K_3}{1+PQ}$; $\overline{N_3}=K_3$ (4)

With the terms and conditions $K_1 > PK_2 + \alpha K_3$ *and* $K_3 > K_1/\alpha Q$

3.Analysis of Stability at Co-Existent State

Lemma (3.1):

$$If\ \Delta = \begin{bmatrix} \frac{-a_{11}(K_1-\alpha K_3-PK_2)}{1+PQ} & \frac{-a_{11}P(K_1-\alpha K_3-PK_2)}{1+PQ} & -a_{11}\frac{\alpha(K_1-\alpha K_3-PK_2)}{1+PQ} \\ \frac{a_{22}Q(QK_1+K_2-Q\alpha K_3)}{1+PQ} & \frac{-a_{22}(QK_1+K_2-Q\alpha K_3)}{1+PQ} & 0 \\ o & 0 & -K_3a_{33} \end{bmatrix}$$

Then the system is stable only when $(\alpha_1+\beta)^2 > 4\alpha_1\beta(1+pq)$, $(\alpha_1+\beta)^2=4\alpha_1\beta(1+PQ)$ $(\alpha_1+\beta)^2 < 4\alpha_1\beta(1+pq)$

Proof: The characteristic roots are :

$$\lambda = -k_3a_{33},\ \lambda=\frac{-(\alpha_1+\beta)\pm\sqrt{(\alpha_1+\beta)^2-4\alpha_1\beta(1+pq)}}{2} \text{ where}$$

$$\alpha_1=\frac{a_{11}(K_1-\alpha K_3-PK_2)}{1+PQ},\beta=\frac{a_{22}(QK_1+K_2-QAK_3)}{1+PQ} \quad (5)$$

Case (i): When $(\alpha_1+\beta)^2 > 4\alpha_1\beta(1+pq)$

In this Case(i), real and negative roots are obtained. Hence, It is stable.

Case(ii): When $(\alpha_1+\beta)^2=4\alpha_1\beta(1+PQ)$

In this case(ii), real and negative roots are found. Therefore, It is stable.

Case (iii): When $(\alpha_1+\beta)^2 < 4\alpha_1\beta(1+pq)$

Here, the roots which are complex with negative real part occurred, thus the state is stable.

Lemma(3.2):

If Δ be the jocobian matrix (mentioned above) then the solution curves are representing asymptotical stability.

Proof: The solution curves are obtained as

$$U_1 = \frac{(\lambda_1+\beta)[U_{10}(\lambda_1+K_3a_{33})-\alpha\alpha_1U_{30}]-\alpha_1PU_{20}(\lambda_1+K_3a_{33})}{(\lambda_1-\lambda_2)(\lambda_1+K_3a_{33})}e^{\lambda_1 t}$$

$$+\frac{(\lambda_2+\beta)[U_{10}(\lambda_2+K_3a_{33})-\alpha\alpha_1U_{30}]-\alpha_1PU_{20}(\lambda_2+K_3a_{33})}{(\lambda_2-\lambda_1)(\lambda_2+K_3a_{33})}e^{\lambda_2 t}-\frac{(\beta-K_3a_{33})\alpha\alpha_1U_{30}}{(\lambda_1+K_3a_{33})(\lambda_2+K_3a_{33})}e^{-K_3a_{33}t}$$

$$U_2 = \frac{(\lambda_1+\alpha_1)(\lambda_1+\beta)(\lambda_1+K_3a_{33})+(\lambda_1+K_3a_{33})\beta QU_{10}-\beta Q\alpha\alpha_1U_{30}}{(\lambda_1-\lambda_2)(\lambda_1+K_3a_{33})}e^{\lambda_1 t}$$

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$$+\frac{(\lambda_2+\alpha_1)(\lambda_2+\beta)(\lambda_2+K_3a_{33})+(\lambda_2+K_3a_{33})\,\beta\,Qu_{10}-\alpha\,\alpha_1\beta Q\,u_{30}}{(\lambda_2-\lambda_1)(\lambda_2+K_3a_{33})}e^{\lambda_2 t}-\frac{\alpha\,\alpha_1\beta Q\,u_{30}}{(\lambda_1+K_3a_{33})(\lambda_2+K_3a_{33})}e^{-K_3a_{33}t}$$

$$U_3 = U_{30}\ e^{-K_3a_{33}t} \qquad (2.4)$$

Geometrical interpretation:

The system is asymptotically stable in the three cases (i). $U_{10} > U_{20} > U_{30}$, (ii). $U_{10} > U_{30} > U_{20}$ & (iii). $U_{20} > U_{30} > U_{10}$. The solution curves are illustrated in Fig.1 to Fig.3

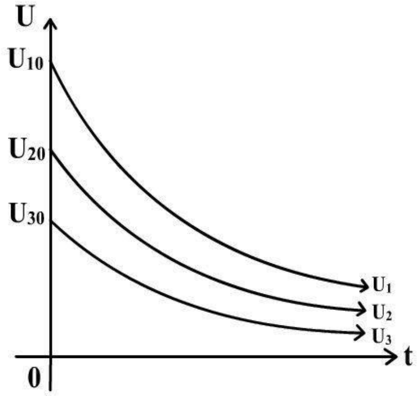

Fig.1: Case (i): $U_{10} > U_{20} > U_{30}$

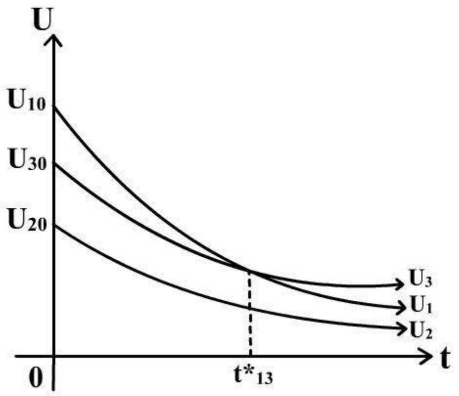

Fig.2: Case (ii): $U_{10} > U_{30} > U_{20}$

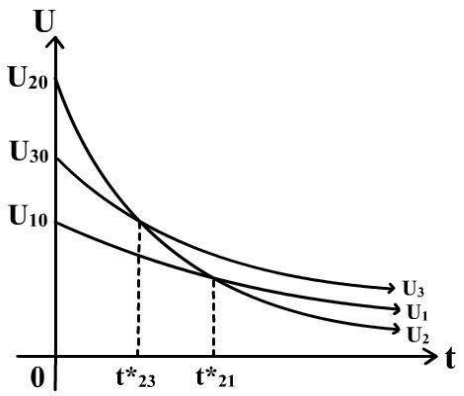

Fig.3: Case (iii): $U_{20} > U_{30} > U_{10}$

4. Analysis of Stability in Different Aspects

Stability Analysis in terms of Local and Global Stabilities of considered significant ecosystem by R-H criteria, Lyapunov Theorem, Diffusive analysis, Stochastic Analysis is discussed with the following theorems.

Theorem (4.1): *If* $\Delta = \begin{bmatrix} -a_{11}\overline{N_1} & -a_{12}\overline{N_1} & -a_{13}\overline{N_1} \\ a_{21}\overline{N_2} & -a_{22}\overline{N_2} & 0 \\ 0 & 0 & -a_{33}\overline{N_3} \end{bmatrix}$ is *a* Jocbian matrix with the valid conditions

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$K_1-\overline{N_1}-P\overline{N}_2-\alpha\overline{N_3}=0, K_2-\overline{N}_2+Q\overline{N_1}=0, K_3-\overline{N_3}=0$ then the corresponding system is Locally stable at Coexistence Equilibrium State $E^*\left(\overline{N_1},\overline{N_2}\right)$

Proof:

The characteristic equation of A is $|\Delta-\lambda I|=0$

i.e $\begin{vmatrix} -a_{11}\overline{N_1}-\lambda & -a_{12}\overline{N_1} & -a_{13}\overline{N_1} \\ a_{21}\overline{N_2} & -a_{22}\overline{N_2}-\lambda & 0 \\ 0 & 0 & -a_{33}\overline{N_3}-\lambda \end{vmatrix}=0$

$$\lambda=\left(-a_{33}\overline{N_3}-\lambda\right)\left[\left(\lambda+a_{11}\overline{N_1}\right)\left(\lambda+a_{22}\overline{N_2}\right)+a_{12}a_{21}\overline{N_1}\,\overline{N_2}\right]=0$$

$$i.e\ \lambda=\lambda^3+\lambda^2\left(a_{11}\overline{N_1}+a_{22}\overline{N_2}+a_{33}\overline{N_3}\right)+\lambda\left[\overline{N_1}\,\overline{N_2}\left(a_{11}a_{22}+a_{12}a_{21}\right)+a_{33}\overline{N_3}\left(a_{11}\overline{N_1}+a_{22}\overline{N_2}\right)\right.$$
$$\left.+a_{33}\overline{N_1}\,\overline{N_2}\,\overline{N_3}\left(a_{11}a_{22}+a_{12}a_{21}\right)\right]$$

By arranging Routh Array, all the elements in the first column are positive.

Those are $1>0$, $X_1>0, \frac{X_1X_2-1.X_3}{X_1}>0$ & $X_3>0$ where $X_1=a_{11}\overline{N_1}+a_{22}\overline{N_2}+a_{33}\overline{N_3}$

$X_2=\overline{N_1}\,\overline{N_2}\left(a_{11}a_{22}+a_{12}a_{21}\right)+a_{33}\overline{N_3}\left(a_{11}\overline{N_1}+a_{22}\overline{N_2}\right)$, $X_2=a_{33}\overline{N_1}\,\overline{N_2}\,\overline{N_3}\left(a_{11}a_{22}+a_{12}a_{21}\right)$

$$\frac{X_1X_2-1.X_3}{X_1}=\frac{1}{a_{11}\overline{N_1}+a_{22}\ \overline{N_2}+a_{33}\ \overline{N_3}}\left\{\left[a_{11}{}^2a_{22}\ \ \overline{N_1}+a_{11}a_{12}a_{21}\overline{N_1}+a_{11}a_{22}{}^2\overline{N_2}+a_{12}a_{21}\overline{N_2}\right]\overline{N_1}\,\overline{N_2}\right.$$
$$\left.+\left[a_{11}{}^2a_{33}\overline{N_1}+a_{11}a_{33}{}^2\overline{N_3}\right]\overline{N_1}\,\overline{N_3}+\left[a_{22}{}^2a_{33}\overline{N_2}+a_{22}a_{33}{}^2\overline{N_3}\right]\overline{N_2}\,\overline{N_3}+2a_{11}a_{22}a_{33}\overline{N_1}\,\overline{N_2}\,\overline{N_3}\right\}>0$$

The system is therefore locally stable under the R-H criterion at $E^*(N_1, N_2)$

Theorem(4.2): The positive equilibrium $E^*\left(\overline{N_1},\overline{N_2},\overline{N_3}\right)$ of system (2.1)-(2.3) is globally asymptotically stable only when $V(t)=\left(N_1-\overline{N_1}-\overline{N_1}\ln\left(\frac{N_1}{\overline{N_1}}\right)\right)+l_1\left(N_2-\overline{N_2}-\overline{N_2}\ln\left(\frac{N_2}{\overline{N_2}}\right)\right)+l_2\left(N_2-\overline{N_2}-\overline{N_2}\ln\left(\frac{N_2}{\overline{N_2}}\right)\right)(4.0)$, *where* $l_1,l_2>0$.

Proof: The suitable Lyapunov function $V(t)$ is defined as (4.0) for verifying the global stability at the interior equilibrium point $E^*\left(\overline{N_1},\overline{N_2},\overline{N_3}\right)$.

$$\frac{dV}{dt}=\left(\frac{N_1-\overline{N_1}}{N_1}\right)\frac{dN_1}{dt}+l_1\left(\frac{N_2-\overline{N_2}}{N_2}\right)\frac{dN_2}{dt}+l_2\left(\frac{N_3-\overline{N_3}}{N_3}\right)\frac{dN_3}{dt}$$

$$\frac{dV}{dt}=(N_1-\overline{N_1})\,(a_1-a_{11}N_1-a_{12}N_2-a_{13}N_1N_3)+l_1(N_2-\overline{N_2})\,(a_2-a_{22}N_2-a_{21}N_1)$$

$$+l_2(N_3-\overline{N_3})\,(a_3-a_{33}N_3)$$

$$\frac{dV}{dt}\leq-\left[(a_{11}+\frac{a_{12}}{2}+\frac{a_{13}}{2}+\frac{a_{21}}{2a_{22}})(N_1-\overline{N_1})^2+(a_{22}+\frac{a_{12}}{2}+\frac{a_{21}}{2a_{22}})(N_2-\overline{N_2})^2+(a_{33}+\frac{a_{13}}{2})(N_3-\overline{N_3})^2\right]\Rightarrow\frac{dV}{dt}<0$$

Thus $V'(t)<0$, The non-diffusive system (2.1)-(2.2) is further globally asymptotically stable by the theorem of Lyapunov.

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Theorem (4.3):

The diffusive equation system is constituted as

$$\frac{\partial N_1}{\partial t} = a_1 N_1 - a_{11} N_1^{\,2} - a_{12} N_1 N_2 - a_{13} N_1 N_3 + D_1 \frac{\partial^2 N_1}{\partial u^2} \qquad (4.1)$$

$$\frac{\partial N_2}{\partial t} = a_2 N_2 - a_{22} N_2^{\,2} - a_{21} N_2 N_1 + D_2 \frac{\partial^2 N_2}{\partial u^2} \qquad (4.2)$$

$$\frac{\partial N_3}{\partial t} = a_3 N_3 - a_{33} N_3^{\,2} + D_3 \frac{\partial^2 N_3}{\partial u^2} \qquad (4.3)$$

where D_1, D_2, D_3 represents the constant diffusion coefficients of N_1,N_2,N_3 respectively

Then all the Eigen values of the system (4.1)-(4.3) are having negative parts if and only if the conditions $A > 0$, $C > 0$, $AB - C < 0$ *hold*

where $A = a_{11}\overline{N_1} + a_{22}\overline{N_2} + a_{33}\overline{N_3} + K^2(D_1 + D_2 + D_3)$;

$B = a_{11}\overline{N_1}a_{22}\overline{N_2} + a_{33}\overline{N_3}a_{22}\overline{N_2} + a_{11}\overline{N_1}a_{33}\overline{N_3} + K^2(a_{22}\overline{N_2}(D_1 + D_3) + a_{11}\overline{N_1}(D_2 + D_3) + a_{33}\overline{N_3}D_1 + D_2))$
$+K^4(D_1 D_3 + D_2 D_3 + D_1 D_2) + a_{12}\overline{N_1}a_{21}\overline{N_2}$;

$C = (a_{33}\overline{N_3} + K^2 D_3)(a_{12}\overline{N_1}a_{21}\overline{N_2}) + (a_{11}\overline{N_1} + K^2 D_1)(a_{22}\overline{N_2} + K^2 D_2)(a_{33}\overline{N_3} + K^2 D_3)$

The set of equations (4.1)-(4.3) is a diffusion system with the conditions on $N_1(\mathrm{u},t)$, $N_2(\mathrm{u},t)$ and $N_3(\mathrm{u},t)$ in $0 \le u \le L, L > 0$ as below

$$\frac{\partial N_1(0,t)}{\partial t} = \frac{\partial N_1(L,t)}{\partial t} = \frac{\partial N_2(0,t)}{\partial t} = \frac{\partial N_2(L,t)}{\partial t} = \frac{\partial N_3(0,t)}{\partial t} = \frac{\partial N_3(L,t)}{\partial t} = 0$$

To discuss the system's steady state, the system(4.1)-(4.3) can be linearized by putting $N_1 = \overline{N_1} + v_1$, $N_2 = \overline{N_2} + v_2$, $N_3 = \overline{N_3} + v_3$

and we obtain

$$\frac{\partial v_1}{\partial t} = -a_{11}\overline{N_1}N_1 - a_{12}\overline{N_1}N_2 - a_{13}\overline{N_1}N_3 + D_1 \frac{\partial^2 N_1}{\partial u^2} \qquad (4.4)$$

$$\frac{\partial v_2}{\partial t} = -a_{22}\overline{N_2}N_2 + a_{21}\overline{N_2}N_1 + D_2 \frac{\partial^2 N_2}{\partial u^2} \qquad (4.5)$$

$$\frac{\partial v_3}{\partial t} = -a_{33}N_3\overline{N_3} + D_3 \frac{\partial^2 N_3}{\partial u^2} \qquad (4.6)$$

Let the solutions of the system (4.4)- (4.6) be in the form of

$v_1(u,t) = \alpha_1 e^{\lambda t}\cos ku$; $v_2(u,t) = \alpha_2 e^{\lambda t}\cos ku$ $v_3(u,t) = \alpha_3 e^{\lambda t}\cos ku$.

Then the model becomes

$$v_1'(t) = -a_{11}\overline{N_1}N_1 - a_{12}\overline{N_1}N_2 - a_{13}\overline{N_1}N_3 + D_1(\mathrm{K}^2\mathrm{N}_1) \qquad (4.7)$$

$$v_2'(t) = -a_{22}\overline{N_2}N_2 + a_{21}\overline{N_2}N_1 + D_2(\mathrm{K}^2\mathrm{N}_2) \qquad (4.8)$$

$$v_3'(t) = -a_{33}N_3\overline{N_3} + D_3(\mathrm{K}^2\mathrm{N}_3) \qquad (4.9)$$

The characteristic equation of the variational matrix of the system (4.7)-(4.9) is in the form of

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \qquad (4.10)$$

where $A = a_{11}\overline{N_1} + a_{22}\overline{N_2} + a_{33}\overline{N_3} + K^2(D_1 + D_2 + D_3)$;

$B = a_{11}\overline{N_1}a_{22}\overline{N_2} + a_{33}\overline{N_3}a_{22}\overline{N_2} + a_{11}\overline{N_1}a_{33}\overline{N_3} + K^2(a_{22}\overline{N_2}(D_1 + D_3) + a_{11}\overline{N_1}(D_2 + D_3) + a_{33}\overline{N_3}D_1 + D_2))$
$+K^4(D_1 D_3 + D_2 D_3 + D_1 D_2) + a_{12}\overline{N_1}a_{21}\overline{N_2}$

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$C=(a_{33}\overline{N_3}+K^2D_3)(a_{12}\overline{N_1}a_{21}\overline{N_2})+(a_{11}\overline{N_1}+K^2D_1)(a_{22}\overline{N_2}+K^2D_2)(a_{33}\overline{N_3}+K^2D_3)$

By the Routh-Hurwitz criterion, all the Eigen values of (4.10) have negative parts if and only if $A>0$, $C>0$, $AB-C>0$.

Theorem(4.4): If the interior equilibrium point $\left(\overline{N_1},\overline{N_2},\overline{N_3}\right)$ of the system without diffusion is globally stable, then the corresponding uniform steady state of the diffusive model (4.1)-(4.3) under zero flux boundary conditions is also globally asymptotically stable.

Proof:- Let us define the function $V_1(t)=\int_0^R V(N_1,N_2,N_3)du$

$V(N_1,N_2,N_3)=\left[(N_1-\overline{N_1})-\overline{N_1}\ln(\frac{N_1}{\overline{\overline{N_1}}})\right]+l_1\left[(N_2-\overline{N_2})-\overline{N_2}\ln(\frac{N_2}{\overline{\overline{N_2}}})\right]+l_2\left[(N_3-\overline{N_3})-\overline{N_3}\ln(\frac{N_3}{\overline{\overline{N_3}}})\right]$ Now we differentiate V_1 w.r.to t along with N_1,N_2,N_3 of the diffusive model (4.1)-(4.3) we get

$V_1^1(t)=\int_0^R\left(\frac{\partial v}{\partial N_1}.\frac{\partial N_1}{\partial t}+\frac{\partial v}{\partial N_2}.\frac{\partial N_2}{\partial t}+\frac{\partial v}{\partial N_3}.\frac{\partial N_3}{\partial t}\right)du=I_1+I_2$

where $I_1=\int_0^R\frac{dv}{dt}du$ and $I_2=\int_0^R\left(D_1\frac{\partial v}{\partial N_1}\frac{\partial^2N_1}{\partial u^2}+D_2\frac{\partial v}{\partial N_2}\frac{\partial^2N_2}{\partial u^2}+D_3\frac{\partial v}{\partial N_3}\frac{\partial^2N_3}{\partial u^2}\right)du$

Using the established result of B.Dubey&J.Hussain [1],

$I_2=-D_1\int_0^R\frac{\partial^2v}{\partial N_1{}^2}\left(\frac{\partial N_1}{\partial u}\right)^2du-D_2\int_0^R\frac{\partial^2v}{\partial N_2{}^2}\left(\frac{\partial N_2}{\partial u}\right)^2du-D_3\int_0^R\frac{\partial^2v}{\partial N_3{}^2}\left(\frac{\partial N_3}{\partial u}\right)^2du$

$I_2=-D_1\int_0^R\frac{\overline{N_1}}{N_1{}^2}\left(\frac{\partial N_1}{\partial u}\right)^2du-D_2\int_0^R\frac{\overline{N_2}}{N_2{}^2}\left(\frac{\partial N_2}{\partial u}\right)^2du-D_3\int_0^R\frac{\overline{N_3}}{N_3{}^2}\left(\frac{\partial N_3}{\partial u}\right)^2du$

It is observed that if $I_1<0$ then $\frac{dV_1}{dt}<0$

Hence globally asymptotically stable of the system is obtained.

5. Stochastic Analysis

The following system of non-linear ordinarily differential equations establishes the model equations for the constructed significant ecosystem as

(i) The rate of growth equation for Prey-Ammensal species (N_1):

$$\frac{\mathrm{dN_1}}{\mathrm{dt}}=a_1N_1-a_{11}N_1^2-a_{12}N_1N_2-a_{13}N_1N_3+\Psi_1\Omega_1(t) \quad (5.1)$$

(ii) The rate of growth equation for predator species (N_2):

$$\frac{\mathrm{dN_2}}{\mathrm{dt}}=a_2N_2-a_{22}N_2^2+a_{12}N_1N_2+\Psi_2\Omega_2(t) \quad (5.2)$$

(iii) The rate of growth equation for enemy species (N_3):

$$\frac{\mathrm{dN_3}}{\mathrm{dt}}=a_3N_3-a_{33}N_3^2+\Psi_3\Omega_3(t) \quad (5.3)$$

Let $\Psi_1,\Psi_2\,\&\,\Psi_3$ are real constants, $\Omega(t)=\left[\Omega_1(t),\Omega_2(t),\Omega_3(t)\right]$ is a 3D Gaussian white noise process satisfying $E\left[\Omega_i(t)\right]=0\quad i=1,2,3$

Where $\delta_{i,j}$ is the Kroneckersymbol ; δ is the $\delta-$dirac function.

$N_1(t)=u_1(t)+s^*, N_2(t)=u_2(t)+p^*\ \&\ N_3(t)=u_3(t)+q^*$

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$$\frac{dN_1}{dt}=\frac{du_1}{dt},\ \frac{dN_2}{dt}=\frac{du_2}{dt},\ \frac{dN_3}{dt}=\frac{du_3}{dt}$$

From equation (5.1)

$$\frac{du_1}{dt}=-a_{13}u_1u_3^*-a_{13}u_1q^*-a_{13}u_3s^*-a_{13}s^*q^*+\Psi_1\Omega_1(t) \quad (5.4)$$

From equation (5.2)

$$\frac{du_2}{dt}==a_2u_2+a_2p^*-a_{22}u_2^2-2a_{22}u_2p^*-a_{22}p^{*2}+a_{21}u_1u_2+a_{21}u_1p^*+a_{21}u_2s^*+a_{21}s^*p^*+\Psi_2\Omega_2(t) \quad (5.5)$$

From equation (5.3)

$$\frac{du_3}{dt}==a_3u_3+a_3q^*-a_{33}u_3^2-2a_{33}u_3q^*-a_{33}q^{*2}+\Psi_3\Omega_3(t) \quad (5.6)$$

The like part of (5.4),(5.5)&(5.6) is given by

$$\frac{du_1}{dt}=-a_{11}u_1s^*-a_{12}u_2s^*-a_{13}u_3s^*+\psi_1\Omega_1(t) \quad (5.7)$$

$$\frac{du_2}{dt}=-a_{22}u_2p^*+a_{21}u_1p^*+\psi_2\Omega_2(t)\ \&\ \frac{du_3}{dt}=-a_{33}u_3q^*+\psi_2\Omega_2(t) \quad (5.8)\&\ (5.9)$$

Taking Fourier technique of (5.7),(5.8)&(5.9),we get

Eq (5.7) $\Rightarrow \psi_1(\omega)\overline{\Omega_1}=\left(i\omega+a_{11}s^*\right)\overline{u_1}(\omega)+a_{12}\,s^*\overline{u_2}(\omega)\ +a_{13}\,s^*\overline{u_3}(\omega)$ (5.10)

Eq (5.8) $\Rightarrow \Psi_2\overline{\Omega_2}(\omega)=\left(i\omega+a_{22}p^*\right)\overline{u_2}(\omega)-a_{21}\,p^*\overline{u_1}(\omega)$ (5.11)

Eq(5.9) $\Rightarrow \Psi_3\overline{\Omega_3}(\omega)=\left(i\omega+a_{33}q^*\right)\overline{u_3}(\omega)$ (5.12)

The matrix form of (5.10),(5.11)&(5.12) is $\Rightarrow M(\omega)\overline{u(\omega)}=\Omega(\omega)$ (5.13)

$$\text{Here } M(\omega)=\begin{bmatrix} M_{11} & M_{12} & M_{13}\\ M_{21} & M_{22} & M_{23}\\ M_{31} & M_{32} & M_{33}\end{bmatrix};\ \bar{u}(\omega)=\begin{bmatrix}u_1(\omega)\\ u_2(\omega)\\ u_3(\omega)\end{bmatrix};\ \bar{\Omega}(\omega)=\begin{bmatrix}\psi_1\bar{\Omega}_1(\omega)\\ \psi_2\bar{\Omega}_2(\omega)\\ \psi_3\bar{\Omega}_3(\omega)\end{bmatrix}$$

$$\text{where } M(\omega)=\begin{bmatrix} -i\omega+a_{11}s^* & a_{12}s^* & a_{13}s^*\\ -a_{21}p^* & i\omega+a_{22}p^* & 0\\ 0 & 0 & i\omega+a_{33}p^*\end{bmatrix}$$

$$\Rightarrow \left|\det M(\omega)\right|=R(\omega)^2+I(\omega)^2$$

$$\Rightarrow \left|\det M(\omega)\right|^2=(-\omega^2 a_{22}\,p^*-\omega^2 a_{11}\,s^*-\omega^2 a_{33}\,q^*+a_{11}a_{22}a_{33}\,s^*p^*q^*+a_{12}a_{21}a_{33}\,s^*p^*q^*)^2$$
$$+(-\omega^3\omega a_{11}a_{22}\,s^*p^*+\omega a_{12}a_{21}\,s^*p^*+\omega a_{22}a_{33}\,p^*q^*+a_{11}a_{33}\,s^*q^*)^2$$

$$\textit{From } (5.13)\Rightarrow \bar{u}(\omega)=\left[M(\omega)\right]^{-1}+\bar{\Omega}(\omega)$$

$$\text{where } \left[M(\omega)\right]^{-1}=k(\omega)=\frac{1}{M(\omega)}\begin{bmatrix} M_{11}^{CF(1,1)^T}(\omega) & M_{21}^{CF(1,2)^T}(\omega) & M_{31}^{CF(1,3)^T}(\omega)\\ M_{12}^{CF(2,1)^T}(\omega) & M_{22}^{CF(2,2)^T}(\omega) & M_{32}^{CF(2,3)^T}(\omega)\\ M_{13}^{CF(3,1)^T}(\omega) & M_{23}^{CF(3,2)^T}(\omega) & M_{33}^{CF(3,3)^T}(\omega)\end{bmatrix}$$

$$\textit{Now } \sigma_{u_1}^2=\frac{1}{2\pi}\sum_{i=1}^{3}\int_{-\infty}^{\infty}\Psi_i\left|\frac{M_{i1}^{CF(2,i)^T}}{|M(\omega)|}\right|^2 d\omega,\ \sigma_{u_2}^2=\frac{1}{2\pi}\sum_{i=1}^{3}\int_{-\infty}^{\infty}\Psi_i\left|\frac{M_{i2}^{CF(2,i)^T}}{|M(\omega)|}\right|^2 d\omega,\ \sigma_{u_3}^2=\frac{1}{2\pi}\sum_{i=1}^{3}\int_{-\infty}^{\infty}\Psi_i\left|\frac{M_{i3}^{CF(3,i)^T}}{|M(\omega)|}\right|^2 d\omega$$

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$$\Rightarrow \left|M_{11}{}^{CF(1,1)^{T}}(\omega)\right|^{2} = (a_{22}a_{33}\,p^{*}q-\omega^{2})^{2}+\omega^{2}(a_{22}\,p^{*}+a_{33}\,q^{*})^{2};$$

$$\left|M_{12}{}^{CF(2,1)^{T}}(\omega)\right|^{2} = (a_{21}a_{33}\,q^{*})^{2}+\omega^{2}(a_{21}\,\omega p^{*})^{2}\,,\left|M_{13}{}^{CF(3,1)^{T}}(\omega)\right|^{2}=0;$$

$$\left|M_{21}{}^{CF(1,2)^{T}}(\omega)\right|^{2} = (a_{12}a_{33}\,s^{*}q^{*})^{2}+\omega^{2}(a_{12}\,\omega s^{*})^{2};$$

$$\left|M_{22}{}^{CF(2,2)^{T}}(\omega)\right|^{2} = (a_{11}a_{33}\,s^{*}q^{*}-\omega^{2})^{2}+\omega^{2}(a_{11}\,s^{*}+a_{33}\,q^{*})^{2}\,;\left|M_{23}{}^{CF(3,2)^{T}}(\omega)\right|^{2}=0;$$

$$\left|M_{31}{}^{CF(1,3)^{T}}(\omega)\right|^{2} = (a_{13}\,\omega s^{*})^{2}+(a_{13}a_{22}\,s^{*}\,p^{*})^{2}\,;\left|M_{32}{}^{CF(2,3)^{T}}(\omega)\right|^{2}=(a_{21}a_{13}\,s^{*}p^{*})^{2};$$

$$\left|M_{33}{}^{CF(3,3)^{T}}(\omega)\right|^{2} = (a_{11}a_{12}\,s^{*}p^{*}+a_{12}a_{21}\,s^{*}p^{*}-\omega^{2})^{2}+\omega^{2}(a_{11}\,s^{*}+a_{22}\,p^{*})^{2}$$

$Case(1):\ If\ \Psi_1=0\ \&\ \Psi_2=0$

$$then\ \ \sigma_{u_1}{}^{2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Psi_3(a_{13}\omega s^{*})^{2}+(a_{13}a_{22}s^{*}\,p^{*})^{2}}{\left|M(\omega)\right|^{2}}d\omega\,,\quad \sigma_{u_2}{}^{2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Psi_3(a_{21}a_{13}s^{*}\,p^{*})^{2}}{\left|M(\omega)\right|^{2}}d\omega$$

$$\sigma_{u_3}{}^{2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Psi_3\left[(a_{11}a_{12}s^{*}\,p^{*}+a_{12}a_{21}s^{*}\,p^{*}-\omega^{2})^{2}+\omega^{2}(a_{11}s^{*}+a_{22}\,p^{*})^{2}\right]}{\left|M(\omega)\right|^{2}}d\omega$$

$Case(2):\ If\ \Psi_1=0\ \&\ \Psi_3=0$

$$then\ \ \sigma_{u_1}{}^{2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Psi_2(a_{12}a_{33}s^{*}\,q^{*})^{2}+(a_{12}\omega s^{*})^{2}}{\left|M(\omega)\right|^{2}}d\omega$$

$$\sigma_{u_2}{}^{2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Psi_2(a_{11}a_{33}s^{*}\,q^{*}-\omega^{2})^{2}+\omega^{2}(a_{11}s^{*}+a_{33}q^{*})^{2}}{\left|M(\omega)\right|^{2}}d\omega\,,\quad \sigma_{u_3}{}^{2}=0$$

$$Case(3):If\ \ \Psi_2=0\ \&\ \Psi_3=0\ then\ \ \sigma_{u_1}{}^{2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Psi_1(a_{22}a_{33}p^{*}\,q^{*}-\omega^{2})^{2}+\omega^{2}(a_{22}p^{*}+a_{33}q^{*})^{2}}{\left|M(\omega)\right|^{2}}d\omega$$

$$\sigma_{u_2}{}^{2}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Psi_1(a_{21}a_{33}\,q^{*})^{2}+(a_{21}\omega p^{*})^{2}}{\left|M(\omega)\right|^{2}}d\omega\,;\quad \sigma_{u_3}{}^{2}=0$$

Clearly the steadiness of populations for smaller estimations of mean square vacillations is pointed out by the population variances.

6. Series Solutions by Homotopy Perturbation Method

The series solutions are obtained by HPM as

$$N_1(t)=c_1+(a_1-a_{11}c_1-a_{12}c_2-a_{13}c_3)\,c_1t$$

$$+\Big[\big(a_1-2a_{11}c_1-a_{12}c_2-a_{13}c_3\big)\,(a_1-a_{11}c_1-a_{12}c_2-a_{13}c_3)\,-a_{12}(a_2-a_{22}c_2+a_{21}c_1)c_2-a_{13}\big(a_3-a_{33}c_3\big)c_3\ \Big]c_1\frac{t^2}{2}+$$

$$\Big\{\big(a_1-2a_{11}c_1-a_{12}c_2-a_{13}c_3\big)c_1\Big[\big(a_1-2a_{11}c_1-a_{12}c_2-a_{13}c_3\big)$$

$$(a_1-a_{11}c_1-a_{12}c_2-a_{13}c_3)-a_{12}(a_2-a_{22}c_2+a_{21}c_1)c_2-a_{13}\big(a_3-a_{33}c_3\big)c_3\Big]$$

$$-2c_1\Big[a_{11}(a_1-a_{11}c_1-a_{12}c_2-a_{13}c_3)c_1+a_{12}(a_2-a_{22}c_2+a_{21}c_1)c_2+a_{13}\big(a_3-a_{33}c_3\big)c_3\ \Big]$$

$$(a_1-a_{11}c_1-a_{12}c_2-a_{13}c_3)-a_{12}c_1c_2\Big[(a_2-2a_{22}c_2+a_{21}c_1)\,(a_2-a_{22}c_2+a_{21}c_1)$$

$$+a_{21}\,(a_1-a_{11}c_1-a_{12}c_2-a_{13}c_3)\,c_1\Big]-a_{13}c_1\big(a_3-2a_{33}c_3\big)\big(a_3-a_{33}c_3\big)c_3\Big\}\frac{t^3}{6}+.....$$

$$N_2(t)=c_2+(a_2-a_{22}c_2+a_{21}c_1)c_2t$$

$$+\big[(a_2-2a_{22}c_2+a_{21}c_1)(a_2-a_{22}c_2+a_{21}c_1)+a_{21}(a_1-a_{11}c_1-a_{12}c_2-a_{13}c_3)c_1\big]c_2\frac{t^2}{2}$$
$$+a_{21}(a_1-a_{11}c_1-a_{12}c_2-a_{13}c_3)c_1\big]+2c_2(a_2-a_{22}c_2+a_{21}c_1)\big[a_{21}c_1(a_1-a_{11}c_1-a_{12}c_2-a_{13}c_3)$$
$$-a_{22}(a_2-a_{22}c_2+a_{21}c_1)c_2\big]+a_{21}c_1c_2\big[(a_1-2a_{11}c_1-a_{12}c_2-a_{13}c_3)(a_1-a_{11}c_1-a_{12}c_2-a_{13}c_3)$$
$$-a_{12}(a_2-a_{22}c_2+a_{21}c_1)c_2-a_{13}c_3\left(a_3-a_{33}c_3\right)\big]\big\}\frac{t^3}{6}+.....$$
$$N_3(t)=c_3+\left(a_3-a_{33}c_3\right)c_3t+\left(a_3-2a_{33}c_3\right)\left(a_3-a_{33}c_3\right)c_3\frac{t^2}{2}$$
$$+\left(a_3-a_{33}c_3\right)c_3\left[\left(a_3-2a_{33}c_3\right)^2-2a_{33}\left(a_3-a_{33}c_3\right)c_3\right]\frac{t^3}{6}+.........$$

7.Conclusions

On the basis of this study, we draw the following conclusions on an important three species significant eco-system:

(i).The asymptotic stability of the system in various cases is asserted by using geometrical interpretation.
(ii).Local stability of the system is observed at interior equilibrium state by Routh-Hurwitz criterion
(iii).Global Stability is also established by constructing suitable Lyapunov function.
(iv).Diffusion and Stochastic Analysis addressed effectively the stability of the system.
(v).Homotopy perturbation method efficiently extracted the series solutions of the three species ecological model.

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