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# A matrix method based on Lucas Polynomials for solving Fractional Differential Equation

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**ABSTRACT**

Our work is categories into two sections. First section concerns with originatingan operational matrix for fractional derivative of Lucas polynomials. With the help of this matrix,a spectral algorithm for solving some fractional-order initial value problems is exhibited and implemented. And the second section deal with a matrix whose entries are the coefficients appearing in the expansion of the derivative of Fibonacci polynomials and whose the sum of the elements in the same row gives the first order derivative of classical Fibonacci sequence at $x = 1$.

**Keywords**
Operational matrix, Fractional differential equation, Fractional Calculus.
**A.M.S. subject classification:**11-XX, 11B39.

## 1. Introduction

Ordinary differential equations involving fractional order derivatives are used to model a variety of systems, of which an important engineering application lies in viscoelastic damping[1].The operational matrices of derivatives are used for solving both linear and non-linear ordinary and fractional differential equations [2].In [3], author concerned with deriving an operational matrix of fractional order derivative of Fibonacci polynomials.
Consider the following fractional differential equation:

$$\begin{cases} D^{\alpha}y(x) + y^{(k)}(x) + y(x) = f(x) \\ y^{(r)}(0) = c_r \end{cases} \qquad (1)$$

where $r = 0,1,2,\ldots,m-1, k = 0,1,2,\ldots,m, m-1 < \alpha \leq m$

### 1.1 Preliminaries and notation

#### 1.1.1 Properties of Fractional Calculus

The fractional derivative of $y(t)$ is understood in the Caputo sense [4]. The caputo derivative has the following property for $n-1 < \alpha < n$

$$D^{\alpha}t^{k} = \begin{cases} \frac{k!}{\Gamma(k-\alpha+1)}t^{k-\alpha}, k \geq \alpha \\ 0, \qquad k < \alpha \end{cases} \qquad (2)$$

#### 1.1.2 Some relevant properties of Lucas Polynomials

The Lucas polynomials is defined by the following recurrence relation

$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x), n \geq 0$$

with initial conditions $L_0\,(x) = 2, L_1(x) = x$, and they can be generated [5] by the following form representation:

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$$L_n(x) = \Sigma_{\mathrm{k}=0}^{\left[\frac{n}{2}\right]} \left(\frac{n}{n-k}\right) \binom{n-k}{k} x^{n-2k} (3)$$

**Theorem1.1.2.1 The following inversion formula to (3) in which the polynomials $x_i$ is expressed in terms of Lucas polynomials is**

$$\boldsymbol{x^n} = \begin{cases} \boldsymbol{\Sigma_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i} L_{n-2i}(x)}, & \boldsymbol{if\ n\ is\ odd} \\ \boldsymbol{\Sigma_{i=0}^{\left[\frac{n-1}{2}\right]} (-1)^i \binom{n}{i} L_{n-2i}(x) + (-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} L_0(x)}, & \boldsymbol{if\ n\ is\ even} \end{cases} \qquad \textbf{(4)}$$

**Proof**

**CaseI: When*n*is odd.**

Clearly, results hold for $n = 1$.Let the result holds for $n$ ,therefore

$$x^n = \Sigma_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i} L_{n-2i}(x)$$

Note that multiplying both sides of the equality by $x$ and using the definition of the Lucas polynomials yields

$$x^{n+1} = \Sigma_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i}\ xL_{n-2i}(x) = \Sigma_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i} \left(L_{n-2i+1}(x) - L_{n-2i-1}(x)\right)$$
$$= \Sigma_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i} L_{n-2i+1}(x) - \Sigma_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i} L_{n-2i-1}(x)$$

Our end goal is to combine these two summations into one. To accomplish this, note that shifting the indices of the second summation up by 1, we have

$$x^{n+1} = \Sigma_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i} L_{n-2i+1}(x) - \Sigma_{i=1}^{\left[\frac{n}{2}\right]+1} (-1)^{i-1} \binom{n}{i-1} L_{n-2i+1}(x) =$$
$$\Sigma_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i} L_{n-2i+1}(x) - \Sigma_{i=1}^{\left[\frac{n}{2}\right]+1} (-1)^{i-1} \binom{n}{i-1} L_{n-2i+1}(x)$$
(5)

In order for our hypothesis to be correct, it suffices to show that

$$x^{n+1} = \Sigma_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n+1}{i} L_{n-2i+1}(x) + (-1)^{\frac{n+1}{2}} \binom{n}{\frac{n+1}{2}} L_0(x)$$

To prove this, we have from (5),

$$x^{n+1} = L_{n+1} + \Sigma_{i=1}^{\left[\frac{n}{2}\right]} (-1)^i \binom{n}{i} L_{n-2i+1}(x) - \Sigma_{i=1}^{\left[\frac{n}{2}\right]} (-1)^{i-1} \binom{n}{i-1} L_{n-2i+1}(x) -$$
$$(-1)^{\frac{n}{2}} \binom{n}{\frac{n-1}{2}} L_0(x)$$
(6)

Now, the coefficient of $L_{n-2i+1}(x) = \binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}$, therefore equation (6) becomes

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$$x^{n+1} = L_{n+1} + \Sigma_{i=1}^{\left[\frac{n}{2}\right]}(-1)^i \binom{n+1}{i} L_{n-2i+1}(x) + (-1)^{\frac{n+1}{2}} \binom{n}{\frac{n+1}{2}} L_0(x)$$
$$= \Sigma_{i=0}^{\left[\frac{n}{2}\right]}(-1)^i \binom{n+1}{i} L_{n-2i+1}(x) + (-1)^{\frac{n+1}{2}} \binom{n}{\frac{n+1}{2}} L_0(x)$$

Which is what we want.

**Case II**When $n$ is even

Let the result holds for $n$ i.e.,

$$x^n = \Sigma_{i=0}^{\left[\frac{n-1}{2}\right]}(-1)^i \binom{n}{i} L_{n-2i}(x) + (-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} L_0(x)$$

Multiplying both sides of the equality by $x$ and using the definition of the Lucas polynomials yields

$$x^{n+1} = \Sigma_{i=0}^{\left[\frac{n-1}{2}\right]}(-1)^i \binom{n}{i} xL_{n-2i}(x) + (-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} xL_0(x)$$
$$= \Sigma_{i=0}^{\left[\frac{n-1}{2}\right]}(-1)^i \binom{n}{i} (L_{n-2i+1}(x) - L_{n-2i-1}(x)) + 2(-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} L_1(x)$$
$$= \Sigma_{i=0}^{\left[\frac{n-1}{2}\right]}(-1)^i \binom{n}{i} L_{n-2i+1}(x) - \Sigma_{i=0}^{\left[\frac{n-1}{2}\right]}(-1)^i \binom{n}{i} L_{n-2i-1}(x) + 2(-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} L_1(x)$$

Our end goal is to combine these two summations into one. To accomplish this, note that shifting the indices of the second summation up by 1, we have

$$x^{n+1} = \Sigma_{i=0}^{\left[\frac{n-1}{2}\right]}(-1)^i \binom{n}{i} L_{n-2i+1}(x) - \Sigma_{i=1}^{\left[\frac{n-1}{2}\right]+1}(-1)^{i-1} \binom{n}{i-1} L_{n-2i+1}(x) + 2(-1)^{\frac{n}{2}} \binom{n-1}{\frac{n}{2}} L_1(x)$$

Since $n$ is even, therefore above becomes

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$$x^{n+1} = \Sigma_{i=0}^{\left[\frac{n-1}{2}\right]}(-1)^i\binom{n}{i}L_{n-2i+1}(x) - \Sigma_{i=1}^{\left[\frac{n}{2}\right]}(-1)^{i-1}\binom{n}{i-1}L_{n-2i+1}(x)$$
$$+2(-1)^{\frac{n}{2}}\binom{n-1}{\frac{n}{2}}L_1(x)$$
$$= L_{n+1} + \Sigma_{i=1}^{\left[\frac{n-1}{2}\right]}(-1)^i\binom{n}{i}L_{n-2i+1}(x) - \Sigma_{i=1}^{\left[\frac{n-1}{2}\right]}(-1)^{i-1}\binom{n}{i-1}L_{n-2i+1}(x)$$
$$-(-1)^{\left[\frac{n}{2}\right]-1}\binom{n}{\left[\frac{n}{2}\right]-1}L_1(x) + 2(-1)^{\frac{n}{2}}\binom{n-1}{\frac{n}{2}}L_1(x)$$
$$= L_{n+1} + \Sigma_{i=1}^{\left[\frac{n-1}{2}\right]}(-1)^i\binom{n+1}{i}L_{n-2i+1}(x) + (-1)^{\left[\frac{n}{2}\right]}\binom{n}{\left[\frac{n}{2}\right]-1}L_1(x)$$
$$+2(-1)^{\frac{n}{2}}\binom{n-1}{\frac{n}{2}}L_1(x)$$
$$= \Sigma_{i=0}^{\left[\frac{n-1}{2}\right]}(-1)^i\binom{n+1}{i}L_{n-2i+1}(x) + (-1)^{\left[\frac{n}{2}\right]}\left(\binom{n}{\left[\frac{n}{2}\right]-1} + \binom{n-1}{\left[\frac{n}{2}\right]}\right)$$
$$= \Sigma_{i=0}^{\left[\frac{n}{2}\right]}(-1)^i\binom{n+1}{i}L_{n-2i+1}(x)$$

which is what we desire.

## 1.2Operational matrix of Derivative

Suppose that the equation (1) has a continuous function solution that can be expressed in the Lucas polynomials $y(x) = \Sigma_{k=1}^{\infty}a_kL_k(x)$, and let $y_n(x)$ be an approximation to $y(x)$,that is

$$y_n(x) = \Sigma_{k=1}^{n+1}a_kL_k(x) = A^TL(x),$$

where $A^T = [a_1a_2 \dots a_{n+1}], L(x) = [L_1(x)L_2(x)\ \dots L_{n+1}(x)]^T$

### 1.2.1 Lucas operational matrix of derivative

The $rth$ order derivative of (1) can be written as [6]

$$y^{(r)}(x) = A^TD^rL(x), D = [d_{ij}] = \begin{cases} i\sin\frac{(j-i)\pi}{2}, & j > i \\ 0, & j \le i \end{cases}, r = 0,1,2,\dots n \quad (7)$$

where $D$ is $n\times n$ operational matrix for the derivative.

The caputo derivative of the vector $F$ can be expressed by

$$D^\alpha L(x) = B^{(\alpha)}L(x) \quad (8)$$

Where $B^{(\alpha)}$ is the $(n+1)\times(n+1)$ operational matrix of fractional derivative. In this section, we derive matrix $B^{(\alpha)}$.Using (2)$and$ (4) for $i = 0,1,2,,\dots,n$

$$D^\alpha L_i(x) = \Sigma_{j=0}^{\left[\frac{i}{2}\right]}\left(\frac{i}{i-j}\right)\binom{i-j}{j}D^\alpha x^{i-2j}$$
$$= \begin{cases}\Sigma_{j=0}^{\left[\frac{i}{2}\right]}\left(\frac{i}{i-j}\right)\binom{i-j}{j}\frac{\Gamma(i+1-2j)}{\Gamma(i+1-2j-\alpha)}x^{i-2j-\alpha}, & i-2j \ge \alpha \\ 0, & i-2j < \alpha\end{cases}$$

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If $i-2j \geq \alpha$, and $i-2j$ is odd , then using (4) ,we have

$$D^{\alpha}L_i(x) = x^{-\alpha}\Sigma_{j=0}^{\left[\frac{i}{2}\right]}\left(\frac{i}{i-j}\right)\binom{i-j}{j}\frac{\Gamma(i+1-2j)}{\Gamma(i+1-2j-\alpha)}x^{i-2j}$$

$$= x^{-\alpha}\Sigma_{j=0}^{\left[\frac{i}{2}\right]}\left(\frac{i}{i-j}\right)\binom{i-j}{j}\frac{\Gamma(i+1-2j)}{\Gamma(i+1-2j-\alpha)}A_{ijk}\,L_{i-2j-2k}$$

$$= x^{-\alpha}\Sigma_{j=0}^{\left[\frac{i}{2}\right]}B_{ijk}\,L_{i-2j-2k}$$

where

$$A_{ijk} = \Sigma_{k=0}^{\left[\frac{i-2j}{2}\right]}(-1)^k\binom{i-2j}{k}, B_{ijk} = \left(\frac{i}{i-j}\right)\binom{i-j}{j}\frac{\Gamma(i+1-2j)}{\Gamma(i+1-2j-\alpha)}A_{ijk}$$

If $i-2j \geq \alpha$, and $i-2j$ is even , then then using (4) ,we have

$$D^{\alpha}L_i(x) = x^{-\alpha}\Sigma_{j=0}^{\left[\frac{i}{2}\right]}\left(\frac{i}{i-j}\right)\binom{i-j}{j}\frac{\Gamma(i+1-2j)}{\Gamma(i+1-2j-\alpha)}x^{i-2j}$$

$$= x^{-\alpha}\Sigma_{j=0}^{\left[\frac{i}{2}\right]}\left(\frac{i}{i-j}\right)\binom{i-j}{j}\frac{\Gamma(i+1-2j)}{\Gamma(i+1-2j-\alpha)}A_{ijk}\,L_{i-2j-2k}$$
$$+ x^{-\alpha}\Sigma_{j=0}^{\left[\frac{i}{2}\right]}\left(\frac{i}{i-j}\right)\binom{i-j}{j}\frac{\Gamma(i+1-2j)}{\Gamma(i+1-2j-\alpha)}(-1)^{\frac{i-2j}{2}}\binom{i-2j-1}{\frac{i-2j}{2}}L_0(x)$$

$$= x^{-\alpha}\Sigma_{j=0}^{\left[\frac{i}{2}\right]}B_{ijk}\,L_{i-2j-2k}$$
$$+ x^{-\alpha}\Sigma_{j=0}^{\left[\frac{i}{2}\right]}\left(\frac{i}{i-j}\right)\binom{i-j}{j}\frac{\Gamma(i+1-2j)}{\Gamma(i+1-2j-\alpha)}(-1)^{\frac{i-2j}{2}}\binom{i-2j-1}{\frac{i-2j}{2}}L_0(x)$$

where

$$A'_{ijk} = \Sigma_{k=0}^{\left[\frac{i-2j-1}{2}\right]}(-1)^k\binom{i-2j}{k}, B'_{ijk} = \left(\frac{i}{i-j}\right)\binom{i-j}{j}\frac{\Gamma(i+1-2j)}{\Gamma(i+1-2j-\alpha)}A'_{ijk}$$

## 1.3Numerical Method

In this section, we describe in detail how the operational matrix of fractional derivative of Lucas polynomials can be employed for solving fractional differential equations. Substituting approximation obtained into (1), we find that

$$\begin{cases} B^{(\alpha)}L(x) + A^T D^k L(x) + A^T L(x) = f(x) \\ A^T D^r L(0) = c_r, r = 0,1, \dots m-1 \end{cases} \tag{9}$$

### 1.3.1 Illustrative Test Problems

In this section we apply the method presented to solve the following test examples.
**Example** Consider the following equation

$$\begin{cases} y^{(3)}(x) + D^{\left(\frac{3}{2}\right)}y(x) + y(x) = 1 + x \\ y(0) = 1 \\ y^{(1)}(0) = 1 \\ y^{(2)}(0) = 0 \end{cases} \tag{10}$$

We applied the Lucas polynomial approach to solve (8) with $n = 2$. In this case we have

$$L = [L_0(x), L_1(x), L_2(x)] = [2\,, x\,, x^2 + 2\ \ ], \qquad A = [a_1, a_2\,, a_3]$$

$$D^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B^{\frac{3}{2}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{\Gamma(3)}{\Gamma\left(\frac{3}{2}\right)}x^{-\frac{3}{2}} & 0 & \frac{\Gamma(3)}{\Gamma\left(\frac{3}{2}\right)}x^{-\frac{3}{2}} \end{bmatrix}$$

And

$$A^T D^3 L + A^T B^{\left(\frac{3}{2}\right)} L + A^T L = f(x),$$

$$y(0) = \sum_{k=0}^{3} a_k L_k(0) = 1, y'(0) = \sum_{k=0}^{3} a_k D L_k(0) = 1, y"(0) = \sum_{k=0}^{3} a_k D^2 L_k(0) = 0$$

On simplifying these, we obtain $a_i's$ and we have $y(x) = x + 1$, which is the exact solution

## 2. Derivative of the Fibonacci polynomials

In this section we shall study a matrix which generates the sequences obtained by deriving the Fibonacci polynomials.In [7], author proved many results related to the derivative of Fibonacci polynomial.

### 2.1 Polynomials obtained by deriving the Fibonacci polynomials

By deriving the Fibonacci polynomials, it is obtained:

$F_1^{'}(x) = 0$

$F_2^{'}(x) = 1$

$F_3^{'}(x) = 2x$

$F_4^{'}(x) = 3x^2 + 2$

$F_5^{'}(x) = 4x^3 + 6x$

$F_6^{'}(x) = 5x^4 + 12x^2 + 3$

$F_7^{'}(x) = 6x^5 + 20x^3 + 12x$

$F_8(x) = 7x^6 + 30x^4 + 30x^2 + 4$

$F_9^{'}(x) = 8x^7 + 42x^5 + 60x^3 + 20x$

$F_{10}^{'}(x) = 9x^8 + 56x^6 + 105x^4 + 60x^2 + 5$

Note first, that the equations for the derivatives of Fibonacci polynomials may be written in matrix form as $F^{'} = BX$, where $F^{'} = [F_1^{'}(x), F_2^{'}(x), F_2^{'}(x), \dots]^t, X = [1, x, x^2, \dots]^t$ and $B$ is the lower triangular matrix with entrances the coefficients appearing in the expansion of the derivatives of Fibonacci polynomials in increasing powers of $x$:

$$B=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 12 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 20 & 0 & 6 & 0 & 0 & 0 & 0 \\ 4 & 0 & 30 & 0 & 30 & 0 & 7 & 0 & 0 & 0 \\ 0 & 20 & 0 & 60 & 0 & 42 & 0 & 8 & 0 & 0 \\ 5 & 0 & 60 & 0 & 105 & 0 & 56 & 0 & 9 & 0 \\ 0 & 30 & 0 & 140 & 0 & 168 & 0 & 72 & 0 & 10 \end{bmatrix}$$

Note that matrix $B$ is invertible.

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