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AS and AS_2-Paracompactness in Topological Spaces

[1]*S.Umamaheswari*, [2]*M.Saraswathi*
[1,2]*Assistant Professor of Mathematics*
K.K.C, P.Velur,Namakkal, Tamil Nadu, India
[1]umamaheswari.maths@gmail.com , [2]msmathsnkl@gmail.com

Abstract—The purpose of this paper is to introduce the two new concepts of $\boldsymbol{AS}$-Paracompact spaces and $\boldsymbol{AS_2}$-Paracompact spaces. Also we have proved that every $\boldsymbol{AS}$-Paracompactness and $\boldsymbol{AS_2}$-Paracompactness has a topological property. Furthermore, we have introduced **AS**-normal and its properties.

Keywords—Angelic spaces, seperable, $\boldsymbol{S}$-Paracompact, $\boldsymbol{S_2}$ –Paracompact, $\boldsymbol{S}$-Normal, $\boldsymbol{AS}$-Paracompact, $\boldsymbol{AS_2}$–Paracompact, $\boldsymbol{AS}$-normal.

MSC (2010): 46A50, 54C10, 54D15, 54D20.

I. INTRODUCTION

In this paper, we introduce two new properties in topological spaces which are AS-paracompactness and AS_2-paracompactness. And also introduced the concept of AS-normal. Diedonne.Jintroduced by [4] Paracompact space. Arhangelskii defined by$C$-paracompact and C_2-paracompact. C-paracompact and C_2-paracompact were studied in [15]. Arhangelskii introduced on 2012 a new weaker version of normality issaid to C-normality [2]. In 2019, S-paracompactness and S_2-paracompactness were studied in [13] and Lutfi Kalantan [8] introduced S-normality. Fermlin. D.H [3] introduced the angelic space. The introduction of an angelic space was a step in the study of compactness. In this paper we are using angelic spaces in S-paracompact, S_2-paracompact and S-normal. Angelic paracompact space represented by $\mathcal{APS}$ and angelic paracompact denoted by $\mathcal{AP}$.

II. PRELIMINARIES

Definition 2.1 :[3]

A TS is said to bean ***angelic***, if for each relatively countably compact subset K of F the following hold: (a) K is relatively compact (b) If $k \in \overline{K}$, then $\exists$a sequence in K that converges to k.

Definition 2.2:[12]

A TS is termed as***seperable*** if it has a countable dense subset.

Definition 2.3:[13]

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Let F be a TS and K be an angelic seperable subspace of F. If there is a bijection mapping $r: F \to G$, G is an paracompact(Hausdorff paracompact) space and the restriction $r|_k : K \to r(K)$ is a homeomorphism, then F is said to be a***S-paracompact(S_2-paracompact) space.***

Definition 2.4:[11]

ATSF is termed asa ***P-space*** if F is T_1 and eachG_δ-set is open.

Definition 2.5:[6]

A TS F is said to be ***locally seperable***, if each element has a seperable open neighbourhood.

Definition 2.6:[14]

A space F is of ***countable tightness*** if for every subset K of F and every$f \in F$ with $f \in K \exists$ a countable subset $L \subseteq F \ni l \in L$

III. AS-PARACOMPACTNESS AND AS_2-PARACOMPACTNESS

Definition 3.1:

Let X be an angelic space and A be an angelic seperable subspace of X. If $\exists$ a bijection mapping $f: X \to Y$,Y is an $\mathcal{APS}$ and the restriction $g|_{\mathcal{A}}: \mathcal{A} \to g(\mathcal{A})$ is a homeomorphism, then $\mathcal{P}$ is said to be an ***AS-paracompact space.***

Definition 3.2:

Let $\mathcal{P}$ be an angelic space and $\mathcal{A}$ be an angelic seperable subspace of $\mathcal{P}$. If there is a bijection mapping $g: \mathcal{P} \to Q$, Q is Hausdorff $\mathcal{APS}$ and the restriction $g|_{\mathcal{A}}: \mathcal{A} \to g(\mathcal{A})$ is a homeomorphism, then $\mathcal{P}$ is said to be an ***AS_2-paracompact space.***

Theorem 3.3:

Let$F$ be an angelic separable but not T_2, then F cannot be an AS_2-paracompact.

Proof.

Suppose F is any separable non-T_2 space. Let F be an AS_2-paracompact. Then $\exists$ a$T_2\mathcal{APS}G$ and a bijective function $r: F \to G \ni r|_K: K \to r(K)$ is a homeomorphism for all angelic separable subspaces$K \subseteq F$. Since F is separable, then $r: F \to G$ is a homeomorphism. But G is Hausdorff, then F is Hausdorff which is a contradiction. Hence F cannot be AS_2-paracompact.

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Example 3.4:

Let $(\mathbb{R}, \mathcal{CF})$ be the finite complement topology defined on the real numbers. Since $(\mathbb{R}, \mathcal{CF})$ is an $\mathcal{AP}$ being an angelic compact, then the identity function $id: (\mathbb{R}, \mathcal{CF}) \to (\mathbb{R}, \mathcal{CF})$ shows that it is anAS-paracompact but not anAS_2-paracompact being angelic separable but not T_2 space.

Example 3.5.

Ω_1 is an AS_2-paracompact space which is not an angelic paracompact. Any angelic separable subspace of Ω_1 is countable. Suppose $K \subset \Omega_1$ is uncountable which implies that K is unbounded in Ω_1. If N is any countable subset of K, then $\exists\ \alpha < \Omega_1 \ni \alpha = Sup\ N$. Thus, $\exists\ \eta \in K \ni \alpha < \eta$. The set $((\alpha, \eta] \cap K)$ is a nonempty open subset of K with $\big((\alpha, \eta] \cap K\big) \cap N = \emptyset$. Thus, K cannot be an angelic separable implying that any angelic separable subspace of Ω_1 is countable. Since Ω_1 is T_2 locally angelic compact, then $\exists$ a one to one continuous function, say r, onto a Hausdorff compact space G. Let $K \subset \Omega_1$ be any angelic separable subspace. Since the closure of any countable set is an angelic compact in Ω_1, then we get that $r|_K: K \to r(K)$ is a homeomorphism implying $r|_K: K \to r(K)$ is a homeomorphism.

Theorem 3.6:

If F is an angelic separable but not an angelic paracompact, then F cannot be an AS-paracompact.

Proof:

Suppose$F$ is any angelic separable non-$\mathcal{APS}$. Let$F$ be anAS-paracompact. Then $\exists$ an$\mathcal{APS}G$ and a bijective function $r: F \to G \ni r|_K: K \to r(K)$ is a homeomorphism for everyangelic separable subspaces$K \subseteq F$. Since F is an angelic separable, then $r: F \to G$ is a homeomorphism. But G is an $\mathcal{AP}$, then F is $\mathcal{AP}$ which is a contradiction. Hence $\mathcal{F}$ cannot be an $\mathcal{AS}$ -paracompact.

Theorem 3.7:

Every $\mathcal{T}_2$ countably compact angelic separable $\mathcal{AS}$ -paracompact space isan angelic compact.

Proof:

Suppose$\mathcal{F}$is any $\mathcal{T}_2$ countably angelic compact angelic separable $\mathcal{AS}$ -paracompact space. Then $\mathcal{F}$ is an $\mathcal{AP}$ because the witness function of $\mathcal{AS}$ -paracompactness is a homeomorphism. Since any countably angelic compact $\mathcal{T}_2\mathcal{APS}$ is an angelic compact. Hence$\mathcal{F}$ is an angelic compact.

Theorem 3.8:

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If F is an $\mathcal{AS}$ -paracompact ($\mathcal{AS}_2$-paracompact) space and Fr´echet such that $r : F \to G$ is a witness of $\mathcal{AS}$ -paracompactness ($\mathcal{AS}_2$-paracompactness) of F, then r is continuous.

Proof:

Suppose F is an $\mathcal{AS}$ -paracompact and Frechet. Let $r : F \to G$ witnesses $\mathcal{AS}$ -paracompactness of F. Let $K \subseteq F$ and pick $g \in r(\bar{K})$ implying that $\exists$ a a unique $f \in F \ni r(f) = g$ and $f \in \bar{K}$. Since F is Frechet, then $\exists$ a sequence $a_n \subseteq K \ni a_n \to f$. The subspace $L = \{f, a_n : n \in N\}$ of F is an angelic separable by being countable, thus $r\,|_L : L \to r(L)$ is a homeomorphism. Now, let $P \subseteq G$ be any open neighborhood of g. Then $P \cap r(L)$ is open in the subspace $r(L) \subseteq g$. Since $r(\{a_n : n \in N\}) \subseteq r(L) \cap r(K)$ and $P \cap r(L) \neq \emptyset, P \cap r(L) \neq \emptyset$. Hence $g \in \overline{r(K)}$, thus $r(\bar{K}) \subseteq \overline{r(A)}$. Therefore, r is continuous.

Theorem 3.9:

$\mathcal{AS}$ -paracompactness ($\mathcal{AS}_2$-paracompactness) is a topological property.

Proof:

Let F be an $\mathcal{AS}$ -paracompact space and H be any TS$\ni$ F is homeomorphic to H. Let r be the function witnessing $\mathcal{AS}$ -paracompactness of F onto an $\mathcal{APSG}$ and $t : F \to G$ be a homeomorphism. Then $r \circ t^{-1} : H \to G$ will be the witness of $\mathcal{AS}$ -paracompactness of H. Hence $\mathcal{AS}$ -paracompactness ($\mathcal{AS}_2$-paracompactness) is a topological property.

Theorem 3.10:

The Cartesian product of two $\mathcal{AS}_2$-paracompact spaces is $\mathcal{AS}_2$-paracompact in case that at least one of them is countably compact and Fr´echet.

Proof:

Let F and H be $\mathcal{AS}_2$-paracompact $\ni$ F is countably compact and Frechet. Let G and $r : F \to G$ be witnesses of $\mathcal{AS}_2$-paracompactness of F. Then r is continuous by Theorem 3.8 implying that G is countably angelic compact. Hence, G is compact. Let G' and $r' : H \to G'$ be witnesses of $\mathcal{AS}_2$-paracompactness of H. Consider the function $t := r \times r' : F \times H \to G \times G'$. Observe that $G \times G'$ is T_2angelic paracompact. Now, let N be any angelic separable subspace of $F \times H$. Therefore, $p_1(N) \subseteq F$ and $p_2(N) \subseteq H$ are both separable subspaces of F and H respectively being continuous images of an angelic separable subspace $N \subseteq F \times H$. Then countable product of an angelic separable spaces is angelic separable, $p_1(N) \times p_2(N)$ is an angelic separable in $F \times H$. Thus, as $N \subseteq p_1(N) \times p_2(N)$, we get that $t\,|_L : L \to t(L)$ is a homeomorphism. Hence the Cartesian product of two $\mathcal{AS}_2$-paracompact spaces is $\mathcal{AS}_2$-paracompact in case that at least one of them is countably compact and Fr´echet.

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Theorem 3.11:

$\mathcal{AS}$ -paracompactness ($\mathcal{AS}_2$-paracompactness) is an additive property.

Proof:

Suppose $\{F_\alpha : \alpha \in \Lambda\}$ is a family of $\mathcal{AS}_2$-paracompact spaces. Hence, for every $\alpha \in \Lambda \exists$ an $\mathcal{APS}$ G_α and a bijection $r_\alpha : F_\alpha \to G_\alpha \ni r_\alpha|_{K_\alpha} : K_\alpha \to r_\alpha(K_\alpha)$ is a homeomorphism for everyangelic separable subspace K_α of F_α. Since angelic paracompactness is an additive property, $\bigoplus_{\alpha\in\Lambda} G_\alpha$ is an angelic paracompact. Define $r : \bigoplus_{\alpha\in\Lambda} F \to \bigoplus_{\alpha\in\Lambda} G_\alpha$ as follows: for each $f \in \bigoplus_{\alpha\in\Lambda} F_\alpha$, there exists a unique $\gamma \in \Lambda$ such that $f \in F_\gamma$ then $r(f) = r_\gamma(f)$. Let K be any angelic separable subspace of $\bigoplus_{\alpha\in\Lambda} F_\alpha$. Write $K = \cup\, \alpha \in \Lambda^* \,(K \cap F_\alpha)$ where $\Lambda^* = \{\alpha \in \Lambda : K \cap F_\alpha, \emptyset\}$. Since K is an angelic separable, then Λ^* is countable and $K \cap F_\alpha$ is an angelic separable in F_α for all $\alpha \in \Lambda^*$. Therefore, $r\alpha\ |_{K\cap X_\alpha} : K \cap F_\alpha \to r_\alpha(K \cap F_\alpha)$ is a homeomorphism for every $\alpha \in \Lambda^*$ implying that $r\ |_K : K \to r(K)$ is a homeomorphism.

Theorem3.12:

Every second countable $\mathcal{AS}_2$-paracompact space is metrizable.

Proof:

Suppose (F, τ) is an $\mathcal{AS}_2$-paracompact second countable space which yields that F is an angelic separable $\mathcal{AS}_2$-paracompact. Then F is T_4 implying that F is regular. Since any second countable T_3 space is metrizable, we get that F is metrizable.

Theorem 3.13:

Every P-space is $\mathcal{AS}_2$-paracompact.

Proof:

Suppose (F, τ) is a P-space. If F is countable, then it is discrete implying that F is an $\mathcal{AS}_2$-paracompact. Assume F is uncountable. Let $K \subseteq F$ be an arbitrary uncountable subset of F and let $N \subset K$ be any countable subset of K. Then N is closed set in K and $K \setminus N$ is a non-empty open set in K with $(K \setminus N) \cap N = \emptyset$. Hence, N cannot be dense in K implying that K cannot be separable. Thus, any angelic separable subspace of F must be countable and hence any angelic separable subspace of F is discrete. Take the identity map $id : (F, \tau) \to (F, N)$. Then $id\ |_K : K \to r(K)$ is a homeomorphism for everyangelic separable subspaces K. Hence, (F, τ) is an $\mathcal{AS}_2$-paracompact.

Example 3.14:

Let $(\mathbb{R}, \mathcal{CC})$, where $\mathcal{CC}$ is the countable complement topology defined on $\mathbb{R}$. Since $(\mathbb{R}, \mathcal{CC})$ is P-space, then by Theorem 3.13, $(\mathbb{R}, \mathcal{CC})$ is $\mathcal{AS}_2$-paracompact. Note function witnessing the $\mathcal{AS}_2$-paracompactness here is the identity taken from $\mathcal{CC}$ to the discrete topology defined on $\mathbb{R}$, write $(\mathbb{R}, \mathcal{D})$. However, $id : (\mathbb{R}, \mathcal{CC}) \to (\mathbb{R}, \mathcal{D})$ is not continuous.

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IV. $\mathcal{AS}$ -NORMAL AND ITS PROPERTIES

Definition 4.1:

A $\mathcal{TSF}$ is termed as an$\mathcal{AS}$ -normal if $\exists$ a normal space G and a bijective function $r : F \to G \ni$ the restriction $r|_K : K \to r(K)$ is a homeomorphism for everyangelic separable subspace $K \subseteq F$.

Example 4.2:

Consider the left ray topology defined on $\mathbb{R}$, $(\mathbb{R}, L)$ such that $L = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, a) : a \in \mathbb{R}\}$. It is an example of $\mathcal{AS}$ -normal space by being a normal space which is not $\mathcal{AS}_2$-paracompact space since it is separable and not T_2 space. In fact, it is not even an $\mathcal{AS}$ -paracompact because it is separable not an$\mathcal{APS}$.

Theorem 4.3:

$\mathcal{AS}$ -normality has a topological property.

Proof:

Suppose$F$ is an $\mathcal{AS}$ -normal space and $F \cong H$ and$G$ is a normal space and $r : F \to G$ be a bijective function $\ni r|_M : M \to r(M)$ is a homeomorphism for everyangelic separable subspace $M$of F. Let $t : H \to F$ be a homeomorphism. Hence G and$r \circ t : H \to G$has the topological property.

Theorem 4.4:

$\mathcal{AS}$ -normality is an additive property.

Proof:

SupposeF_α be an $\mathcal{AS}$ -normal space for every$\alpha \in \Lambda$. Then their sum $\bigoplus_{\alpha\in\Lambda} F_\alpha$is $\mathcal{AS}$ -normal. For every$\alpha \in \Lambda$, pick a normal space G_α and a bijective function $r_\alpha : F_\alpha \to G_\alpha \ni r_{\alpha|M_\alpha} : M_\alpha \to r_\alpha(M_\alpha)$ is a homeomorphism for each angelic separable subspace M_α of F_α. Since G_α is normal for every$\alpha \in \Lambda$, the sum $\bigoplus_{\alpha\in\Lambda} G_\alpha$ is normal. Consider the function sum, $\bigoplus_{\alpha\in\Lambda} r_\alpha$:$\bigoplus_{\alpha\in\Lambda} F_\alpha \to \bigoplus_{\alpha\in\Lambda} G_\alpha$ defined by $\bigoplus_{\alpha\in\Lambda} r_\alpha(f) = r_\beta(f)$ if $f \in F_\beta, \beta \in \Lambda$. Now, a subspace $M \subseteq \bigoplus_{\alpha\in\Lambda} F_\alpha$is an angelic separable if and only if the set $\Lambda_0 = \{\alpha \in \Lambda : M \cap F_\alpha \neq \emptyset\}$ is countable and $M \cap F_\alpha$ is angelic separable in F_α for each $\alpha \in \Lambda_0$. If $M \subseteq \bigoplus_{\alpha\in\Lambda} F_\alpha$ is an angelic separable, then $(\bigoplus_{\alpha\in\Lambda} r_\alpha)|_M$ is a homeomorphism because $r_\alpha\, M \cap F_\alpha$ is a homeomorphism for every$\alpha \in \Lambda_0$.

Theorem 4.5:

If F is an$\mathcal{AS}$ -normal and of countable tightness and $r : F \to G$ witnesses the $\mathcal{AS}$ -normality of F. Then r is continuous.

Proof:

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Suppose that F is an AS -normal and of countable tightness. Let $r : F \to G$ be any witness of the AS -normality of F. Let $K \subseteq F$ be arbitrary. We have $r(\overline{K}) = r(\bigcup_{L \in [K]^{\leq \Omega_0}} \overline{L}) = \bigcup_{L \in [K]^{\leq \Omega_0}} r(\overline{L}) \subseteq \bigcup_{L \in [K]^{\leq \Omega_0}} \overline{r(L)} \subseteq \overline{r(K)}$. Therefore, r is continuous.

Remark:

Any first countable space is Frèchet, any Frèchet space is sequential and any sequential space is of countable tightness in [6] we conclude the following corollary.

Corollary 4.6:

(1) If F is AS -normal first countable and $r : F \to G$ witnesses the AS -normality of F. Then r is continuous.

(2) If F is AS -normal Frèchet and $r : F \to G$ witnesses the AS -normality of F. Thenr$f$ is continuous.

(3) If F is AS -normal sequential and $r : F \to G$ witnesses the AS -normality of F. Then r is continuous.

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